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Luca Salasnich†‡§[+] and Fabio Sattin‖¶[*]

† Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy
‡ Istituto Nazionale di Fisica Nucleare, Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy
§ Istituto Nazionale di Fisica della Materia, Unità di Milano, Via Celoria 16, 20133 Milano, Italy
‖ Dipartimento di Ingegneria Elettrica, Università di Padova, Via Gradenigo 6/a, 35131 Padova, Italy
¶ Istituto Nazionale di Fisica della Materia, Unità di Padova, Corso Stati Uniti 4, 35127 Padova, Italy

Abstract. In this paper we prove a recent conjecture about the convergence of the WKB series for the angular momentum operator. We demonstrate that the WKB algorithm for the angular momentum gives the exact quantization formula if all orders are summed. Finally, we discuss the supersymmetric semiclassical quantum mechanics (SWKB), which gives the correct quantization of the angular momentum at the leading order.

1. Introduction

The semiclassical methods used to solve the Schrödinger problem are of extreme importance in understanding the global behaviour of eigenfunctions and energy spectra, since they allow us to obtain analytic expressions. The leading semiclassical approximation (torus quantization) is just the first term of a certain $\hbar$-expansion, which is called WKB (Maslov and Fedoriuk 1981).

Recently it was observed (Prosen and Robnik 1993, Graffi *et al* 1994, Robnik and Salasnich 1997a—in the following this work will be referred to as I) that the torus quantization generally fails to predict the individual energy levels (and the eigenstates) within a vanishing fraction of the mean-energy level spacing. This conclusion is believed to be correct for general systems, including the chaotic ones. Therefore, a systematic study of the accuracy of semiclassical approximation is very important, especially in the context of quantum chaos (Casati and Chirikov 1995, Gutzwiller 1990). Since this is a difficult task, it has been attempted for simple systems, where in a few cases even exact solutions may be found (Dunham 1932, Bender *et al* 1977, Voros 1983, Robnik and Salasnich 1997a).

Robnik and Salasnich (1997b) (this work will be referred to as II) dealt with the WKB expansion for the Kepler problem: it was proved that an exact result is obtained once all terms are summed. In particular, the torus quantization (the leading WKB term) of the

full problem is exact, even if the individual torus quantization of the angular momentum and of the radial Kepler problem separately are not, because the quantum corrections (i.e. terms higher than the torus quantization) compensate mutually term by term. In the paper II Robnik and Salasnich conjectured about the higher terms of the WKB expansion. This conjecture is perfectly reasonable but not rigorously proved. In this work our aim is to prove that the same result of II can be reached rigorously by means of a slightly modified procedure.

In the framework of the supersymmetric semiclassical quantization (SWKB), Comtet *et al* (1985) obtained at the leading order the exact quantization of the radial part of the Kepler problem by using the correct value $L^2 = \hbar^2 l(l+1)$. In the last section we complete the result of Comtet *et al* (1985). In fact, we also show that the exact quantization of the angular momentum is obtained at the first order of the SWKB expansion.

2. Eigenvalue problem for the angular momentum

The eigenvalue equation of the angular momentum operator (Landau and Lifshitz 1977) is

$$\hat{L}^2 Y(\theta, \phi) = \lambda^2 \hbar^2 Y(\theta, \phi) \tag{1}$$

with

$$\hat{L}^2 = \hat{P}_\theta^2 + \frac{\hat{P}_\phi^2}{\sin^2(\theta)} = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot(\theta) \frac{\partial}{\partial \theta} \right) - \hbar^2 \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}. \tag{2}$$

After the substitution

$$Y(\theta, \phi) = T(\theta) e^{im\phi} \tag{3}$$

we obtain

$$T''(\theta) + \cot(\theta) T'(\theta) + \left(\lambda^2 - \frac{m^2}{\sin^2(\theta)} \right) T(\theta) = 0. \tag{4}$$

We shall consider the azimuthal quantum number m as fixed. As is well known, equation (4) is exactly solvable. Its eigenvalues and eigenfunctions can be found in any text on quantum mechanics (see, e.g. Landau and Lifshitz 1977): the former are $\lambda^2 = l(l+1)$, $l \geqslant m$; the latter are the associated Legendre polynomials.

The WKB expansion for equation (4) has been studied in II; it was shown that higher-order terms quickly increase in complexity. The method of solution is to find an analytical recursive expression for all the higher-order terms, to sum the entire infinite series, and show that it is convergent to the exact result. Instead of the original function T we shall use the associated function F:

$$T(\theta) = \frac{F(\theta)}{\sqrt{\sin(\theta)}} \tag{5}$$

from which we obtain

$$F''(\theta) + \left[\left(\lambda^2 + \frac{1}{4} \right) + \frac{1}{\sin^2(\theta)} \left(\frac{1}{4} - m^2 \right) \right] F(\theta) = 0. \tag{6}$$

This equation has the standard form of the one-dimensional Schrödinger equation with $\hbar = 2M \equiv 1$. Its eigenvalues are $(\lambda^2 + \frac{1}{4})$. We make the substitution of variable: $x = \theta + \pi/2$, and the positions $U = m^2 - \frac{1}{4}$, $E = \lambda^2 + \frac{1}{4}$. Then equation (6) becomes

$$-F''(x) + \frac{U}{\cos^2(x)} F(x) = E F(x). \tag{7}$$

This is the main result of our paper, because the problem of the WKB quantization of equation (7) has already been dealt with in I. As we shall show, from I we can prove that: (i) equation (7) can be solved exactly; (ii) a semiclassical expansion of (7) may be carried on to all orders (i.e. all terms may be exactly and analytically computed and summed); (iii) the exact and the semiclassical eigenvalues are the same.

3. WKB series for the angular momentum

We observe that in equation (7) $\hbar$ does not appear, therefore an expansion in powers of this parameter is not possible. To override this difficulty a small parameter ϵ is introduced:

$$-\epsilon^2 F''(x) + \frac{U}{\cos^2(x)} F(x) = E F(x). \tag{8}$$

This parameter ϵ, which will be set to 1 at the end of the calculation, has formally the same role of $\hbar$ as ordering parameter. It has already been used in II to deal with the WKB expansion of (4). The formal WKB expansion for F reads:

$$F(x) = \exp\left(\frac{i}{\epsilon} \sum_{n=0}^{\infty} \sigma_n(x) \epsilon^n\right) \tag{9}$$

and we obtain a recursion relation for the phases:

$$(\sigma_0'(x))^2 = E - \frac{U}{\cos^2(x)} \tag{10}$$

$$\sum_{k=0}^{n} \sigma_k'(x) \sigma_{n-k}'(x) + \sigma_{n-1}''(x) = 0 \qquad n > 0. \tag{11}$$

The quantization condition is obtained by requiring that the wavefunction be single valued:

$$\oint d\sigma = \sum_{k=0}^{\infty} \oint d\sigma_k = 2\pi n_\theta \tag{12}$$

where n_θ is an integer number. All odd terms higher than the first vanish when integrated along the closed contour since they are exact differentials (Bender *et al* 1977)

$$\oint d\sigma_{2k+1} = 0 \qquad k > 0. \tag{13}$$

It may be proved by induction (see I) that the solution of (10), (11) is

$$\sigma_n'(x) = (\sigma_0')^{1-3n} P_n(\cos(x)) \sin^{f(n)}(x) \tag{14}$$

with $f(n) = 0$ for n even, $f(n) = 1$ for n odd,

$$P_n(\cos(x)) = \sum_{l=0}^{g(n)} C_{n,l} \cos^{2l-3n}(x) \tag{15}$$

with $g(n) = (3n - 2)/2$ for n even, $g(n) = (3n - 3)/2$ for n odd, $C_{0,0} = 1$, $C_{1,0} = U/2$, $C_{2k,0} = (-1)^k (U/2)^{2k} \binom{\frac{1}{2}}{k}$ and $C_{2k+1,0} = 0$, $k > 0$. It is not necessary to know the value of the other coefficients since all the terms proportional to $C_{n,l}$, $l > 0$ disappear after integration. The integral (12) becomes (see I for more details)

$$\oint d\sigma = \oint d\sigma_0 + \oint d\sigma_1 + \sum_{k>0}^{\infty} \oint d\sigma_{2k} \tag{16}$$

$$= 2\pi(\sqrt{E} - \sqrt{U}) - \pi - \sum_{k>0}^{\infty} \frac{1}{2}\binom{\frac{1}{2}}{k}\frac{2\pi}{\left(\sqrt{4U}\right)^{2k-1}} \tag{17}$$

$$= 2\pi\left(\sqrt{E} - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{1}{2}\binom{\frac{1}{2}}{k}\frac{2\pi}{\left(\sqrt{4U}\right)^{2k-1}}\right) = 2\pi n_\theta \tag{18}$$

but $\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}x^{1-2k} = \sqrt{1+x^2}$ for $|x| > 1$, therefore equation (18) reads

$$\sqrt{E} - \tfrac{1}{2}\sqrt{1+4U^2} - \tfrac{1}{2} = \sqrt{\lambda^2 + \tfrac{1}{4}} - \tfrac{1}{2}\sqrt{4m^2} - \tfrac{1}{2} = n_\theta. \tag{19}$$

Now, because $E = \lambda^2 + \frac{1}{4}$ and $U = m^2 - \frac{1}{4}$, we obtain

$$\lambda^2 = (m + n_\theta + \tfrac{1}{2})^2 - \tfrac{1}{4} = (m + n_\theta)(m + n_\theta + 1) \tag{20}$$

and, with the position $l = n_\theta + m$, we have

$$\lambda^2 = l(l+1) \tag{21}$$

which is the expected result. Please note that the WKB series is convergent for $|x| > 1$, thus for $m > 0$.

We observe that the ϵ-expansion is equivalent to the $1/U$-expansion (this is clear from the structure of equation (8)). In the limit $U \to \infty$ it is easy to get the WKB expansion to the first order, which gives $\lambda^2 = (l + \frac{1}{2})^2$, i.e. the torus quantization of the angular momentum (Langer 1937).

4. SWKB quantization of the angular momentum

To perform the SWKB of equations (4) or (6), it is necessary to know the ground-state wavefunction $T_0(\theta) = \sin^m(\theta)$ and its eigenvalue $\lambda_0 = m(m+1)$. Then we can define the supersymmetric (SUSY) potential

$$\Phi(\theta) = -\frac{\mathrm{d}\ln(F_0(\theta))}{\mathrm{d}\theta} = -(m + \tfrac{1}{2})\cot(\theta) \tag{22}$$

with

$$F_0(\theta) = T_0(\theta)\sqrt{\sin(\theta)}. \tag{23}$$

From Φ the two SUSY partner potentials and Hamiltonians may be defined

$$H_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + V_\pm(\theta) \tag{24}$$

$$V_\pm(\theta) = \Phi^2(\theta) \pm \Phi'(\theta). \tag{25}$$

It is possible to prove (see Junker 1996 and references therein for details) that: (i) the ground-state energy of H_-, E_-^0, vanishes; (ii) all other eigenvalues of H_-, E_-, coincide with that of H_+; (iii) the spectrum of H_- and that of (7) differ by a constant:

$$E_- = \lambda^2 + \tfrac{1}{4} - (\lambda_0^2 + \tfrac{1}{4}) \tag{26}$$

where $\lambda_0 = m(m+1)$ is the eigenvalue of the ground state of equation (4).

Now we apply the SWKB formalism to H_- of equation (24). At the leading order one gets

$$\int_a^b \sqrt{E_- - \Phi^2(x)}\,\mathrm{d}x = n_\theta\pi \tag{27}$$

with a, b roots of

$$E_- - \Phi^2(x) = 0. \tag{28}$$

This formula is also referred to as the CBC formula, from Comtet *et al* (1985). We observe that on the left-hand side of the previous formulae Φ^2 appears instead of the full potential V_-.

From equations (22), (27) and (28) one easily finds

$$\sqrt{E_- + (m + \tfrac{1}{2})^2} - (m + \tfrac{1}{2}) = n_\theta \tag{29}$$

with $b = -a = \arctan\sqrt{\frac{(m+\frac{1}{2})}{E_-}}$. By inverting the previous formula we have

$$E_- = (n_\theta + m + \tfrac{1}{2})^2 - (m + \tfrac{1}{2})^2 \tag{30}$$

and, by using equation (26) with $\lambda_0 = m(m+1)$, we obtain

$$\lambda^2 = (n_\theta + m)(n_\theta + m + 1) \tag{31}$$

which yields the exact quantization, after the position $l = n_\theta + m$.

5. Conclusions

The three-dimensional central potentials are fundamental in physics, and the semiclassical treatment of them has implications in many fields: factorization properties of the one-dimensional potentials (Infeld and Hull 1957), general properties of the semiclassical quantization of the systems with more than one degree of freedom, both integrable or not. Nevertheless, until the paper of Robnik and Salasnich (1997b), no detailed study had been done on half of the problem, the WKB quantization of the angular part. This paper completes that work because it gives a rigorous proof of the convergence of the WKB series to the exact result. Moreover, in the last section, we have demonstrated that, by using SUSY quantum mechanics, the eigenvalue problem of the angular momentum operator can be solved exactly at the lowest order within the semiclassical approximation.

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